Can the Fundamental Group of a Space be the Rationals?

Fanxin Wu

November 4, 2021

It is a theorem of [Shelah 1988] that for a path-connected, locally path-connected compact metric space X, $\pi_1(X)$ is either finitely generated or uncountable.

- Hatcher, *Algebraic Topology*

Outline

What Shelah actually proved: Suppose X is a path-connected, locally path-connected compact metric space. If X is semi-locally simply connected, then $\pi_1(X)$ is finitely generated; if not, then roughly speaking $\pi_1(X)$ contains a perfect set.

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The first part is purely topological. The second part follows from:

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Shelah[1] originally proved this by forcing and absoluteness argument. Pawlikowski[2] gave an easier proof using elementary descriptive set theory.

Definition 1

A topological space X is *path-connected* if for every two points x, y, there exists $f : [0, 1] \to X$ s.t. $f(0) = x$ and $f(1) = y$. It is called *locally path-connected* if path-connected open sets form a basis.

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Definition 2

Fix a base point $x \in X$. A *loop* is an $f : [0, 1] \to X$ s.t. $f(0) = f(1) = x$. Two loops f, g are *homotopic* if there exists $H : [0, 1] \times [0, 1] \to X$ s.t. $H(s, 0) = f(s)$, $H(s, 1) = g(s)$, and $H(0, t) = H(1, t) = x$ for all $t \in [0, 1]$.

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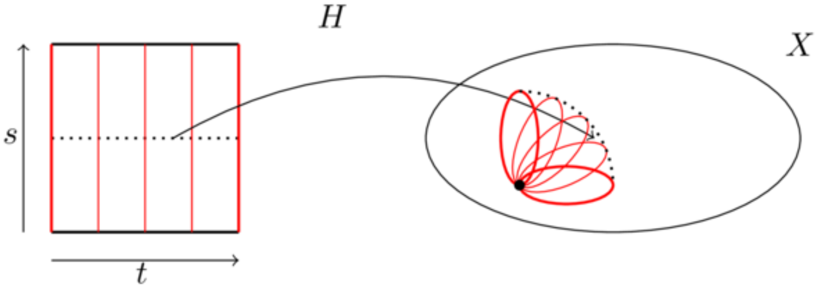

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$$f \cdot g(s) = \begin{cases} f(2s), 0 \leq s \leq 1/2 \\ g(2s-1), 1/2 \leq s \leq 1 \end{cases}$$

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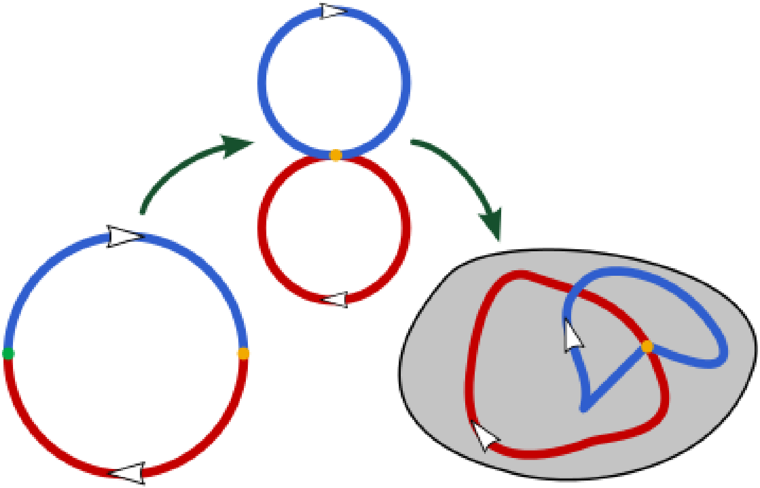
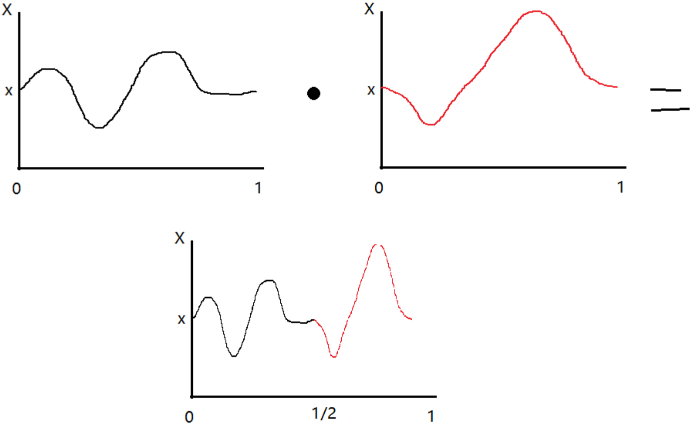
A continuous map $\Phi : X \to Y$ induces a group homomorphism $\Phi_* : \pi_1(X) \to \pi_1(Y), [f] \mapsto [\Phi \circ f]$.

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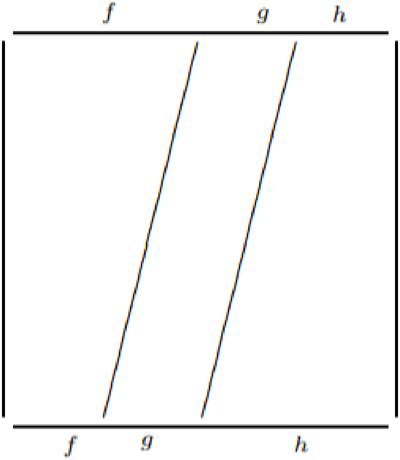
Associativity: $f \cdot (g \cdot h) \neq (f \cdot g) \cdot h$, but they are homotopic.



Examples

$\pi_1(\mathbb{R}^n) = e$, $\pi_1(S) = \mathbb{Z}$, $\pi_1(S \vee S) = \mathbb{Z} * \mathbb{Z}$.

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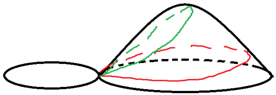
Attaching 2-cells to $\bigvee_{\alpha \in I} S_\alpha$ can create any desired fundamental group, the resulting CW complex is often not metrizable.



Definition 3

X is called *simply connected* if $\pi_1(X)$ is trivial. X is *semi-locally simply connected* if each point has a neighborhood U s.t. $i_* : \pi_1(U) \to \pi_1(X)$ is trivial, i.e., any loop in U can be homotoped to the constant map in X. Note that if $V \subseteq U$ then $i_* : \pi_1(V) \to \pi_1(X)$ is also trivial.

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Examples of slsc spaces: manifolds, CW complexes.
Non-example: The Hawaiian earring $X = \bigcup_n C_n$, C_n the circle at $(\frac{1}{n}, 0)$ of radius $\frac{1}{n}$.

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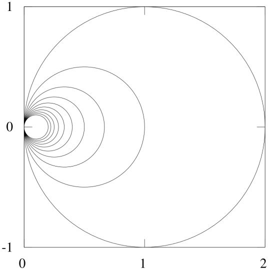
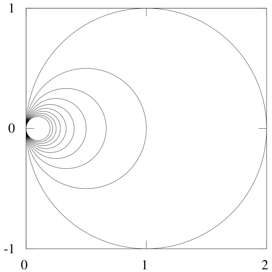
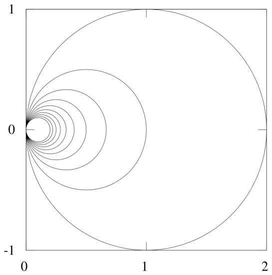




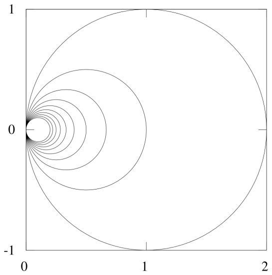
$\pi_1(X)$ is very complicated. For example an $f : [0,1] \to X$ can traverse C_n during $[1 - \frac{1}{2^n}, 1 - \frac{1}{2^{n+1}}]$. It is continuous at 1 since the circles are shrinking.



Let $f_n : [0,1] \to X$, then f is the limit of

$f_1,\ f_1 \cdot f_2,\ f_1 \cdot (f_2 \cdot f_3),\ f_1 \cdot (f_2 \cdot (f_3 \cdot f_4))$...

We may call it $f_1 f_2 f_3 f_4 \cdots$



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We may call it $f_1 f_2 f_3 f_4 \cdots$ Its inverse is $\cdots \bar{f}_4 \bar{f}_3 \bar{f}_2 \bar{f}_1$.

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Each point x has a path-connected open neighborhood U_x s.t. $\pi_1(U_x) \to \pi_1(X)$ is trivial. By compactness we can find a finite cover $\pi_1(U_i) \to \pi_1(X)$. If moreover $\pi_1(U_i \cup U_j) \to \pi_1(X)$ is trivial for any pair $U_i \cap U_j \neq \emptyset$, then we are done: pick $x_i \in U_i$ and for any $U_i \cap U_j \neq \emptyset$ a path in $U_i \cup U_j$ connecting x_i and x_j, then $\pi_1(X)$ is generated by the loops formed using these paths.

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Pick another cover consisting of path-connected open sets of diameter at most $\delta/2$, and apply the argument to this cover.

Proof of second part

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Let U_n be the ball at x with radius $1/n$ and f_n a loop at x that is contained in U_n and nontrivial.

For $c \in 2^{\mathbb{N}}$, define $f_c : [0,1] \to X$ as follows. Let $f_c(1) = x$, and

$$f_c(s) = \begin{cases} f_n(2^{n+1}s - 2^{n+1} + 2) & c(n) = 1 \\ x & c(n) = 0 \end{cases}$$

for $s \in [1 - \frac{1}{2^n}, 1 - \frac{1}{2^{n+1}}]$.

$c \mapsto f_c$ is a continuous map from $2^{\mathbb{N}}$ to $C([0,1], X)$.

Say $c_1 = (1, 1, 0, 1, 0, 1, 0, 0, 1, ...)$ and $c_2 = (1, 1, 1, 1, 0, 1, 0, 0, 1...)$. Then $f_{c_1} = f_0 f_1 e f_3 e f_5 e e f_8 \cdots$ and $f_{c_2} = f_0 f_1 f_2 f_3 e f_5 e e f_8 \cdots$.

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\end{aligned}$$

Since $\pi_1(X)$ is a group and $[f_2] \neq e$, $[f_{c_1}] \neq [f_{c_2}]$.

Define an equivalence relation E on $2^{\mathbb{N}}$:

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E is analytic since $c_1 E c_2 \Leftrightarrow \exists H \in C([0,1] \times [0,1], X)$ such that...

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Theorem

If X is a perfect Polish space and $R \subseteq X^2$ is meager, then there exists a Cantor set $C \subseteq X$ s.t. $\neg xRy$ for any different $x, y \in C$.

For a proof see Theorem 19.1 of Kechris[3].

Proof of proposition

Otherwise, by the 100% lemma (since analytic sets are Baire measurable) E is comeager in some nonempty open subset of $2^{\mathbb{N}} \times 2^{\mathbb{N}}$. Then E is comeager in some basic open set $N_t \times N_s$.

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By Kuratowski-Ulam, E_x is comeager for a comeager set G of x. If G contains x, y that differ at exactly one digit then we are done, since $E_x \cap E_y$ is comeager so nonempty, and we would have xEy, a contradiction.

Lemma

If $G \subseteq 2^{\mathbb{N}}$ is comeager, then it contains some x, y that differ at exactly the first digit.

Proof.

Let $i : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ be the map that flips the first digit. Then i is a homeomorphism. Since G is comeager, so is $i(G)$. If $x \in G \cap i(G)$, then $x, i(x) \in G$ and differ exactly at the first digit. $\square$

Reference

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